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Remarks on a Paper of Passow*

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Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n$, and let

$$f_i: \mathbb{R} \to \mathbb{R}, \qquad f_i(x) = x^{\alpha_i}, \quad i = 1, \ldots, n.$$

Passow [2] proved the following result.

THEOREM 1. *The linear hull of $f_1, \ldots, f_n$ is a Haar space on $\mathbb{R}$ if and only if $\alpha_{i+1} - \alpha_i$ is odd for $i = 1, 2, \ldots, n-1$.*

In this note we shall show that this theorem, and more general results can easily be derived from some general facts about Haar spaces.

Our results are:

THEOREM 2. *If $r_1, \ldots, r_n$ are real polynomials forming a Markov system on $\mathbb{R}$ (i.e., $r_1, \ldots, r_i$ span an i-dimensional Haar space for $i = 1, \ldots, n$) and $r_1 = 1$, then* $\deg r_{i+1} - \deg r_i$ *is positive and odd for $i = 1, \ldots, n-1$.*

COROLLARY. *If U is an n-dimensional Haar space of real polynomials on $\mathbb{R}$ then U has a (Markov) basis $r_1, \ldots, r_n$ such that* $\deg r_1$ *is even, and* $\deg r_{i+1} - \deg r_i$ *is positive and odd for $i = 1, \ldots, n-1$.*

The above-mentioned results on Haar spaces are:

(1) let I be a (finite or infinite) interval, $a \in I$ fixed, $U \subset C(I)$ an $(n-1)$-dimensional Haar space, and $g \in C(I)$ a strictly monotonous function. Then

$$V := \left\{ h \in C(I) \,\middle|\, h(x) = \int_a^x u(t)\, dg(t) + \alpha \text{ for some } \alpha \in \mathbb{R} \text{ and } u \in U \right\}$$

is an n-dimensional Haar space on I,

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(2) let I be an open interval and $U \subset C(I)$ an n-dimensional Haar space. Then there are i-dimensional Haar spaces U_i, $i = 1,..., n$, with $U_1 \subset U_2 \subset \cdots \subset U_n = U$ (i.e., U has a Markov basis);

(3) let I be an open interval and let $U_i \subset C(I)$ be i-dimensional Haar spaces, $i = 1,..., n$, with $U_1 = \{\text{constant functions}\} \subset U_2 \subset \cdots \subset U_n$. Let $f_2 \in U_2 \setminus U_1$ be fixed. Then for all $a \in I$ and $f \in U_n$ there exists

$$(D_1 f)(a) := \lim_{x \to a} \frac{f(x) - f(a)}{f_2(x) - f_2(a)},$$

and $D_1 U_n$ is an $(n - 1)$-dimensional Haar space on I (not necessarily consisting of continuous functions).

(1) is well known (see Schönhage [3, p. 170]).

(2) was proved by Nemeth [1] for finite intervals, and generally by Zielke [4].

(3) is a special case of Theorem 2 in Zielke [5].

Proofs. The sufficiency part of Theorem 1 follows by induction if in (1) we let $g = f_2$ and U be spanned by functions u_i with $u_i(x) = x^{i-2}$, $i = 2,..., n$. The necessity part of Theorem 1 is implied by the corollary.

For reasons of presentation we shall prove Theorem 2 and the corollary for rational functions on $\mathbb{R}$ rather than for polynomials. In this context, a rational function $r = p/q$, where p, q are polynomials, $p \neq 0$, has degree $\deg r = \deg p - \deg q$.

We need some lemmas on rational functions which are fairly obvious.

LEMMA 1. *If r and s are rational functions, we have* $\deg (r/s) = \deg r - \deg s$.

LEMMA 2. *If r is a rational function with* $\deg r \neq 0$, *for the derivative r' we have* $\deg r' = \deg r - 1$.

LEMMA 3. *If r is a nonconstant rational function with* $\deg r \leq 0$, *we have* $\deg r' \leq -2$.

For $n = 1$ Theorem 2 is trivial.

$n - 1 \Rightarrow n$. As r_1 and r_2 span a Haar space, r_2 is strictly monotonous on $\mathbb{R}$. So $\deg r_2$ is odd, and moreover, $\deg r_2 > 0$, for otherwise $\lim_{x \to -\infty} r_2(x) = \lim_{x \to \infty} r_2(x)$. By Lemma 2 we have $\deg r_2' = \deg r_2 - 1 \geq 0$.

By (3), the functions $D_1 r_2, D_1 r_3,..., D_1 r_n$ form a Markov system on $\mathbb{R}$, and $D_1 r_2 = 1$.

By induction hypothesis the differences deg $D_+ r_{i+1}$ − deg $D_+ r_i$ are positive and odd for $i = 2, \dots, n-1$. As $D_+ r_i = r_i'/r_2'$ for all i, we have that $\deg(r_{i+1}'/r_2') - \deg(r_i'/r_2')$ is positive and odd for $i = 2, \dots, n-1$, and so $\deg r_{i+1}' - \deg r_i'$ is positive and odd for $i = 2, \dots, n-1$ by Lemma 1.

Since all r_i', $i \geq 2$, have nonnegative degree, none of the functions r_i, $i \geq 2$, has degree zero because of Lemma 3. From Lemma 2 we conclude $\deg r_{i+1} - \deg r_i = \deg r_{i+1}' - \deg r_i'$ for $i = 2, \dots, n-1$.

To prove the corollary, we apply (2) and obtain a Markov basis $r_1, \dots, r_n$. Then divide by r_1, apply Theorem 2 and Lemma 1.

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